1. Introduction

Deep-hole drilling has found wide application in such fields as metallurgical industry, nuclear power and ordnance industry. However, owing to the working environment of drilling shaft in long and narrow hole, the motion state of observing the drilling shaft during machining is impossible. Therefore, more and more attention has been paid to how to establish the practical model of drilling shaft system to calculate the movement trajectories of the drilling shaft efficiently without losing dynamic analysis precision due to complexity of the problem in the last 20 years [1–5].

At present, much research work has been done on the field of the drilling shaft dynamic in the world. Chin et al. [6,7] presented the problem of state monitoring in deep-hole drilling by computer simulation and experiment. Perng and Deng [8,9] established the different models for lateral and longitudinal shaft motion to study the tool eigenproperties of deep-hole drilling, such as numerical Euler–Bernoulli beam model, dynamic flexibility rotating beam model, and flexibility rotating beam model with static fluid. Such simplified models do not accurately represent the practical drilling shaft system. Drilling shaft system as a rotating machine is a typical nonlinear dynamic system, and hence nonlinear hydrodynamic forces of cutting fluid do not have analytical formulation in fact. Kong et al. [10] established the concentration mass dynamic model of drilling shaft system with the multi-degrees but without considering the effects of inertia and shear, and proposed a method to calculate the nonlinear hydrodynamic forces of cutting fluids and their Jacobian matrices of compatible accuracy simultaneously, and in addition, it was found that the mass eccentricity can inhibit the whirling motion of drilling shaft reported from the experimental results. Hussien et al. [11] studied dynamic drilling shaft modeling with one-span and developed a mathematical approach to study the whirling motion of a continuous boring bar-workpiece system by transforming the problem into nonhomogeneous equations with homogenous boundary conditions. In reality, the drilling shaft system with one or more intermediate supports is typically a multi-part continuum. Despite the cutting forces and the hydrodynamic forces of cutting fluid acting on a few nodal points of drilling shaft individually, the effect from the nonlinearity is global. Additionally, local nonlinearities and linear components of the drilling shaft system are coupled.

For chatter detection, Weinert et al. [12] used dynamical systems to model the drilling process. They are interested in a local description of adequate accuracy to predict disturbances sufficiently and to provide insights into how to react in order to prevent them. Therefore, they proposed a phenomenological approach with a special emphasis on the temporal neighborhoods of instabilities or state transitions from stable drilling to chatter vibration and back. Messaoud [13] proposed a phenomenological model based on the Van Der Pol equation and used nonlinear time series modeling to set up an on-line modeling approach of the time varying dynamics process.
On the other hand, the theories are mentioned in a series of studies concerning the dynamic responses to rotating shaft with various considerations such as subject to shear deformation, rotatory inertia moment, gyroscopic moment, etc. by using Timoshenko shaft model [14–16]. Pan et al. [17] proposed a method for dynamic simulation of multi-body systems including large-scale finite element models of flexibility. Some optimal lumped inertia techniques are developed in order to avoid computation of the coupling matrices between the rigid-body degrees-of-freedom (DOF) and flexible DOF in the finite element representation of the flexible bodies [18–20]. Hu et al. [21] established the finite element model coupling the flexible-body dynamics of a rotating shaft. The flexible DOF were presented as solid elements without modal reduction. However, using the aforementioned method, the numerical analysis of dynamic responses to a nonlinear model with many degrees of freedom generally needs much computing time and may cause computational problems. In order to save computing cost and obtain good accuracy, Fey et al. [22] made a research on the long-term behavior of the mechanical system with local nonlinearity by using the component mode synthesis technique. Hu et al. extended their study to formulate the dynamic responses to a rotating shaft supported by a flexible support structure, and the Craig–Bampton method was employed to reduce the equations of motion [23].

Since drilling shaft system in deep hole machining coupled with the cutting force, the hydrodynamic forces of cutting fluid and the mass eccentricity, nonlinear dynamic behaviors of drilling shaft have rarely been discussed in particularly when they are also with one or more intermediate supports. The present work is a major extension to our previous studies [10], and a system method is developed to satisfy the dynamic design requirement of the large-scale complex deep hole drilling machine. This paper is organized as follows. In Section 2, the governing equations, including the effect of inertia and shear, are formulated as lots of 2-node Timoshenko shaft element model with eight degrees of freedom [24], considering the effects of inertia and shear as shown in Fig. 3. Using the finite element method, flexible drilling shaft equations of lateral motion can be written as

\[ M\ddot{X} + G\dot{X} + KX = F_u + F_c + F_d(X, \dot{X}) \]  

(1)

where \( M, G, K \in \mathbb{R}^{n \times n} \) are the mass matrix, gyroscopic matrix and stiffness matrix respectively, and their specific forms are listed in Appendix A; \( X(t) \in \mathbb{R}^n \) is the displacement vector. For a drilling shaft with \( n \) nodal points, the displacement vector is of the form

\[ X = [x_1, y_1, \psi_1, \ldots, x_n, y_n, \psi_n, \theta_n]^T \]

(2)

where \( x_i, y_i \) and \( \psi_i, \theta_i \) (\( i = 1, 2, \ldots, n \)) are the lateral translations and rotation angles of \( i \)th nodal point along the horizontal and vertical direction, respectively.

Unbalance forces: \( F_u \in \mathbb{R}^{n} \) is the unbalance forces vector caused by the mass eccentricities of drilling shaft and weight forces acting on oxy plane, and its form

\[
F_u = \begin{cases} 
\frac{m^2 e_\omega \omega^2 \cos \omega t}{2} + \frac{m e_\omega \omega^2 \sin \omega t}{2} + m \omega^2 & \\
0 & \\
\vdots & \\
\frac{m^2 e_\omega \omega^2 \cos \omega t}{2} - \frac{m e_\omega \omega^2 \sin \omega t}{2} + m \omega^2 & \\
0 & \\
\vdots & 
\end{cases}
\]

(3)

Fig. 1 displays the boring trepanning association (BTA) drilling machine. The machine has an external cutting fluid supply and internal chip transport. The tool head is screwed onto the drilling tube. The high-pressure cutting fluid is supplied through the space between the drilling tube and machine hole, and then removed along with the chip through the drill tube. The cross-section of the drilling shaft is round.

The drill shaft is considered as a continuous flexible shaft at the shaft driver while the workpiece is fastened at its end. Hence, a typical drilling shaft system with two intermediate simple supports is shown in Fig. 2. The drilling shaft is modeled as lots of 2-node Timoshenko shaft element model with eight degrees of freedom [24], considering the effects of inertia and shear as shown in Fig. 3. Using the finite element method, flexible drilling shaft equations of lateral motion can be written as

Fig. 1. Configuration of deep hole drilling machine for non-rotary workpiece.

Fig. 2. Coordinates of drilling shaft system for calculation.

Fig. 3. The finite element model of drilling shaft.
where \( m_r \) is the element mass, \( e_{x_i} \) and \( e_{y_i} \) \((i = 1, 2, ..., n)\) are mass eccentricities of the drilling shaft acting on the ith nodal point in x and y directions, \( \omega \) is the rotating speed of the drilling shaft, and \( g \) is the acceleration of gravity.

**Cutting forces:** \( F_s \in \mathbb{R}^n \) is the cutting forces vector. The problem that we focus on takes the lateral motion of drilling shaft into consideration. In Ref. [10], the feed rate is assumed as a smallest constant to establish the cutting forces model, and this model reported by the experimental results is feasible. Based on above model, the force terms transmitted to the cutting head of drilling shaft equal to a period lateral fluctuation of a dynamic origin in terms of Eq. (4). The force equals to the value of amplitude \( (f_{co}) \) multiplied by the rotational frequency

\[
\begin{align*}
  f_{cx} &= f_{co} \sin \omega t \\
  f_{cy} &= f_{co} \cos \omega t
\end{align*}
\]

(4)

Thus, the cutting forces vector is of the form

\[
F_c = [0, \ldots, -f_{cx}, -f_{cy}, 0, 0]^T
\]

(5)

where \( f_{cx} \) and \( f_{cy} \) are the fluctuation of cutting forces acting on the cutting head in the x and y directions respectively.

**Hydrodynamic forces of cutting fluid:** \( f^h(X, \dot{X}) \in \mathbb{R}^n \) is the nonlinear hydrodynamic forces vector caused by cutting fluid. Due to the nonlinear hydrodynamic forces acting on the ith nodal point inside drilling hole, thus for drilling shaft system with the length of drilling depth \( (l_l) \), the nonlinear forces vector is of the spatially localized feature as follows

\[
f^h(X, \dot{X}) = [0, \ldots, -f^h_x, -f^h_y, 0, 0, \ldots, -f^h_x, -f^h_y]^T
\]

(6)

where \( f^h_x \) and \( f^h_y \) \((i = 1, 2, ..., n)\) are nonlinear hydrodynamic forces of cutting fluid in the x and y directions respectively, and the computational method of hydrodynamic forces and their Jacobians are considered in Ref. [10].

**Boundary condition:** The boundary condition of clamp side (spindle box side in Fig. 2) of drilling shaft is assumed to be fixed (namely, \( x(z = 0) = y(z = 0) = 0 \) and \( \dot{x}(z = 0) = \dot{y}(z = 0) = 0 \) and that of intermediate support side is assumed to be simply supported (namely, \( x(z = l_{sa}) = 0, y(z = l_{sa}) = 0, x(z = l_{sa} + l_{sb}) = 0 \) and \( y(z = l_{sa} + l_{sb}) = 0 \)).

2.2. Reduction of the dynamic equations of shaft system

The numerical analysis of the dynamic behaviors of drilling shaft system, consisting of linear components with many degrees of freedom and local nonlinearities, needs much computing time and may cause computational problem generally. To save computing cost, the Eq. (1) can be partitioned as

\[
\begin{align*}
  \begin{bmatrix}
    M_{sa} & M_{sb} \\
    M_{sb} & M_b
  \end{bmatrix}
  \begin{bmatrix}
    \dot{X}_a \\
    \dot{X}_b
  \end{bmatrix}
  &=
  \begin{bmatrix}
    G_{sa} & G_{sb} \\
    -G_{sb} & G_b
  \end{bmatrix}
  \begin{bmatrix}
    X_a \\
    X_b
  \end{bmatrix}
  +
  \begin{bmatrix}
    K_{sa} & K_{sb} \\
    K_{sb} & K_b
  \end{bmatrix}
  \begin{bmatrix}
    X_a \\
    X_b
  \end{bmatrix}
  +
  \begin{bmatrix}
    Q_a \\
    Q_b
  \end{bmatrix}
  +
  \begin{bmatrix}
    f_c^h(X, \dot{X}) \\
    0
  \end{bmatrix}
  \end{align*}
\]

(7)

where \( Q = F_c + F_c \). The nonlinear degrees of freedom \( X_a \) \((X_a \in \mathbb{R}^n)\) are caused by nonlinear equations. The linear degrees of freedom \( X_b \) \((X_b \in \mathbb{R}^n)\) depend on \( X_a \) \((n_b > n_a)\).

For reducing the degrees of freedom of the linear components in Eq. (7), \( X \) is written as follows

\[
X = T_1 \eta^T
\]

(8)

where

\[
\eta^T = [\eta_a, \eta_b]^T
\]

\[
T_1 = [\Phi_a, \Phi_b]
\]

(9)

In Eqs. (9) and (10), \( \eta^T = [\eta_1^T, \eta_2^T, \ldots, \eta_m^T] \) and \( \eta_b^T = [\eta^T_{n_a+1}, \eta^T_{n_a+2}, \ldots, \eta^T_m]^T \). The column of matrix \( \phi_a \in \mathbb{R}^{n \times n_a} \) with kept angular eigenfrequencies lower than or equal to \( \omega_{cut} \) [25]. The columns of the matrix \( \phi_b \in \mathbb{R}^{n \times n_b} \) with the residual flexibility modes are defined in the following

\[
\phi_a = [K^{-1} - \phi_b \Omega_{c2} \phi_b^T]^{-1} \left[ \begin{array}{c}
  I_{n_a}
  \\
  0
\end{array} \right]
\]

(11)

where \( I_{n_a} \in \mathbb{R}^{n_a \times n_a} \), \( 0_{n_a} \in \mathbb{R}^{n_b \times n_a} \) are the identity matrix and the zero matrix respectively. The matrix \( \Omega_{c2} \in \mathbb{R}^{n_n \times n_a} \) is a diagonal matrix with kept angular eigenfrequencies lower than or equal to \( \omega_{cut} \) [26]. Then

\[
X = T_1 \eta^T \Rightarrow \left[ \begin{array}{c}
  X_a \\
  X_b
\end{array} \right] = \left[ \begin{array}{c}
  \phi_a \Phi_a \Phi_b \phi_b^T \left[ \begin{array}{c}
  \eta_a \\
  \eta_b
\end{array} \right]
\end{array} \right]
\]

(12)

where

\[
\eta_a = \left[ \begin{array}{c}
  \eta_a^T \\
  0
\end{array} \right], \quad \eta_b = \left[ \begin{array}{c}
  \eta_b^T
\end{array} \right]
\]

(13)

This results in the following total transformation

\[
X = T_1 T_2 \eta = T \eta
\]

(14)

\[
X = T \eta = \left[ \begin{array}{c}
  X_a \\
  X_b
\end{array} \right] = \left[ \begin{array}{c}
  I_{n_a} \\
  \Phi_{ba} \Phi_{ac}^{-1} \Phi_{ac} \left[ \begin{array}{c}
  X_a \\
  X_b
\end{array} \right] + \Phi_{ba} \phi_b \phi_b^T \left[ \begin{array}{c}
  \eta_a \\
  \eta_b
\end{array} \right]
\end{array} \right]
\]

(15)

where \( \eta_b \in \mathbb{R}^m \). It is evident in Eq. (15) that the number of nonlinear degrees of freedom \( X_a \) is not changed, and so the influences are wholly retained.

Applying the transformation Eq. (15), the reduced component equations become

\[
T^T M T \dot{\eta} + T^T \Gamma T \eta + T^T K T \eta = T^T Q + T^T f^h
\]

(16)

After reduction, the \( (n = n_a + n_b) \) order equations of the drilling shaft system are reduced to \( (m = n_a + n_b) \) order equations. The hydrodynamic forces of cutting fluid, cutting forces and unbalance force can be easily added to the reduced governing equations. From Eqs. (12)–(16), it is evident that the external force vectors of the drilling shaft and nonlinear effects definitely remain in the reduced Eq. (16).

The equation of motion of the reduced shaft system is given by

\[
M^* \ddot{\eta} + C^* \dot{\eta} + K^* \eta = F
\]

(17)

where

\[
M^* = T^T M, \quad C^* = T^T C, \quad K^* = T^T K, \quad \text{and} \quad F = T^T Q + T^T f^h
\]

(18)

When state variables \( \dot{\eta} = \dot{\eta}^T \), \( \eta^T = [q_1, q_2, \ldots, q_{2m}]^T \) are introduced, the corresponding system equations in state space are written as Eq. (18), making integration of the system response easy to be done by modified traditional shooting method

\[
\dot{\eta} = \left[ \begin{array}{c}
  M^{-1} (F - G^* \eta - K^* \eta)
\end{array} \right]
\]

(19)
3. Integration of the reduced model equations by modified the traditional shooting method

The transient responses to the reduced system should be integrated step by step numerically. In order to guard the generality of the proposed method, Eq. (18) can be written as follows

\[ \dot{\mathbf{q}} = \mathbf{M}^{-1}(\mathbf{F} - \mathbf{G}\mathbf{q} - \mathbf{K}\mathbf{q}) \Rightarrow \dot{\mathbf{q}} = \mathbf{f}(\mathbf{q}, T, \alpha) \]  

(19)

where \( \tau \) is time. \( \alpha \) is the design parameter of drilling shaft.

It is supposed that a periodic trajectory of the drilling shaft system in Eq. (19) exists and its period is \( T \). In order to show the period \( T \) explicitly in the equation, the system in (19) is transformed into Eq. (20) by using \( \tau = T \tau' \) and then

\[ \frac{d\mathbf{q}}{dT} = T \mathbf{f}(\mathbf{q}, T, \alpha) \]  

(20)

Obviously, the period of the periodic trajectories of the system in Eq. (20) is changed to 1.

The initial values of \( \mathbf{q}^0 = r_i \) and \( T^0 \) \((i = 1, 2, \ldots, 2m)\) are selected, where \( \mathbf{q}^0 = [\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_{2m}]^T \) denotes the state vector at time \( \tau = 0 \). Using the initial values \( \mathbf{q}^0 \) and \( T^0 \), we integrate the system in Eq. (20) from \( \tau = 0 \) to \( \tau = 1 \). Then, the value of \( \mathbf{q}^1 \) is obtained, in which \( \mathbf{q}^1 = [\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_{2m}] \) denotes the state vector at \( \tau = 1 \). Obviously, the value of \( \mathbf{q}^1 \) depends on the initial values, and \( \mathbf{q}^1 \) is a function of \( \beta = [r_1, r_2, \ldots, r_{2m}]^T \) and \( T^0 \), namely \( \mathbf{q}^1(\beta, T^0) \).

A periodic trajectory is found if the following criterion is satisfied.

\[ r_i(\beta, T^0) = q^1_i(\beta, T^0) - q^0_i = q^1_i(\beta, T^0) - r_i = 0 \]  

(21)

where \( r_i \) \((i = 1, 2, \ldots, 2m)\) is the value error. If the chosen initial values of \( q^0_i \) and \( T^0 \) just satisfy the above criterion, the periodic trajectory of the system is found. In other words, the selected initial values of \( q^0_i \) and \( T^0 \) are the exact solutions. However, it is almost impossible in practice. To obtain the initial values of \( q^0_i \) and \( T^0 \), the traditional shooting method is modified by the iteration procedure in detail as follows.

Step 1: The value error \( r_i(\beta, T^0) \) is calculated based on Eq. (22) after integrating from \( \tau = 0 \) to \( \tau = 1 \).

\[ \begin{align*}
   r_1(\beta, T^0) &= q^1_1(\beta, T^0) - r_1 \\
   r_2(\beta, T^0) &= q^1_2(\beta, T^0) - r_2 \\
   \vdots & \\
   r_{2m}(\beta, T^0) &= q^1_{2m}(\beta, T^0) - r_{2m}
\end{align*} \]  

(22)

Step 2: In order to obtain the iteration increment during the iteration process, \( r_i(\beta, T^0) \) is expanded to a Taylor series near to \( r_i \) and \( T^0 \), and retain the linear terms as

\[ \begin{align*}
   r_1 &= \frac{\partial r_1}{\partial \beta_i} \Delta \beta_i + \frac{\partial r_1}{\partial T} \Delta T^0 + \frac{\partial r_1}{\partial q^i} \Delta q^i + \frac{\partial r_1}{\partial q^j} \Delta q^j - r_i = 0 \\
   r_2 &= \frac{\partial r_2}{\partial \beta_i} \Delta \beta_i + \frac{\partial r_2}{\partial T} \Delta T^0 + \frac{\partial r_2}{\partial q^i} \Delta q^i + \frac{\partial r_2}{\partial q^j} \Delta q^j - r_i = 0 \\
   \vdots \\
   r_{2m} &= \frac{\partial r_{2m}}{\partial \beta_i} \Delta \beta_i + \frac{\partial r_{2m}}{\partial T} \Delta T^0 + \frac{\partial r_{2m}}{\partial q^i} \Delta q^i + \frac{\partial r_{2m}}{\partial q^j} \Delta q^j - r_i = 0
\end{align*} \]  

(23)

Step 3: In Eq. (22), the partial derivative of \( r_i(\beta, T^0) \) with respect to \( r_i \) and \( T^0 \) is found. The following formulas can then be obtained.

\[ \frac{\partial r_i}{\partial \beta_j} = \frac{\partial q^i_1}{\partial q^j_1} \frac{\partial q^j_1}{\partial \beta_j} = \frac{\partial q^i_1}{\partial \beta_j} - \delta_{ij}, \quad j = 1, 2, \ldots, 2m \]  

(24)

By selecting the initial values of \( q^0_i = r_i \) and \( T^0 \), the initial condition in Eq. (31) can be discovered that \( \theta^0 = 0 \) (the Kronecker symbol) and \( \theta^0 = 0 \). Consequently, the values of \( \theta^0_i = (\partial q^i_1/\partial \beta_j) \) and \( \theta^0_j = (\partial q^i_1/\partial T^0) \) can be obtained in Eq. (31) integrating from \( \tau = 0 \) to \( \tau = 1 \). At the same time, the system in Eq. (20) should be integrated during the iteration procedure to calculate the value of \( (\partial f_i(\mathbf{q}, T, \alpha)/\partial q^j_1) \) \((i, j = 1, 2, \ldots, 2m)\) in Eq. (31).

Step 5: Next, the values of \( (\partial q^i_1/\partial \beta_j) \) and \( (\partial q^i_1/\partial T^0) \) calculated above are substituted into Eqs. (24) and (25) in Step 3. Then, the values of \( (\partial r_i/\partial \beta_j) \) and \( (\partial r_i/\partial T^0) \) can be obtained.

Step 6: Then, the values of \( (\partial r_i/\partial \beta_j) \) and \( (\partial r_i/\partial T^0) \) are substituted into Eq. (23) in Step 2. It is noticed that a set of the \( 2m \)-order linear equations with \( 2m+1 \) variables will be formed (\( \partial \beta_1, \partial \beta_2, \ldots, \partial \beta_{2m}, \partial T^0 \)). In order to solving the values of \( \Delta \beta_1, \Delta \beta_2, \ldots, \Delta \beta_{2m}, \partial T^0 \), one variable must be fixed. Hence, the optimal method is depicted in detail as follows. In \( r_i \), if the value of \( r_i \) is the least, the initial value of \( \beta_i \) will be selected as a fixed value because it is the closest to that of the actual periodic trajectory of the system. In other words, the column corresponding to the least \( r_i \) should be removed from the coefficient matrix of Eq. (23) at next iterative process. However, if \( \partial T^0 \) is the least in \( r_i \), the column corresponding to \( \Delta T^0 \) cannot be
removed because the period $T^0$ of the periodic trajectories of the system is certain.

Step 7: Finally, if $r_i(\beta, T^0)$ satisfies the criterion in term of Eq. (21), the contrary transform $\tau = t/T$ can be made. Thus, the periodic trajectory of the reduction system in Eq. (19) can be obtained. Otherwise, let $\phi_i^0 = \beta_i + \Delta \beta_i$ ($i=1$, 2, ..., $2m$) and $T^0 = T^0 + \Delta T^0$, and repeat these procedures from Step 1 until the precision required is satisfied.

4. Validation of the proposed method

To validate the proposed method, a computer program has been developed to compute the nonlinear dynamic behaviors of a drilling shaft system with two intermediate supports. The drilling shaft is discretized into 8 finite elements with 9 nodal points (shown in Fig. 2). The corresponding parameters are as follows: the length of the drilling shaft is $l = 4000$ mm, the inner diameter of the drilling shaft is 19 mm, the external diameter of the drilling shaft is 26 mm, drilling depth is $L_c = 427$ mm, the diameter of drilling hole is 28 mm, mass eccentricities of the drilling shaft acting on the $i$th nodal point inside drilling hole ($e_x = e_y = 11.2$ $\mu$m) and other points outside drilling hole ($e_x = e_y = 10$ $\mu$m) have the same rotating phase angle. Based on definition of cutting forces in Eq. (4), the value of fluctuation amplitude is $f_{c0} = 0.067$ KN and the displacements of intermediate supports are $l_{a0} = l_{b0} = (l - l_c)/3$ in this paper. The specifications of other parameters are given in Appendix B.

**Fig. 4.** The trajectories of the drilling shaft center at #9 station for $\omega = 905$ r/min: (a) comparison of the trajectories of the drilling shaft center by the 12-eigenmodes model and full DOF model; (b) comparison of the trajectories of the drilling shaft center by the modified shooting method and Runge-Kutta method.

**Fig. 5.** The trajectories of the drilling shaft center at #8 station for $\omega = 905$ r/min: (a) comparison of the trajectories of the drilling shaft center by the 12-eigenmodes model and full DOF model; (b) comparison of the trajectories of the drilling shaft center by the modified shooting method and Runge-Kutta method.

**Fig. 6.** The trajectories of the drilling shaft center at #7 station for $\omega = 905$ r/min: (a) comparison of the trajectories of the drilling shaft center by the 12-eigenmodes model and full DOF model; (b) comparison of the trajectories of the drilling shaft center by the modified shooting method and Runge-Kutta method.
By using the presented method, the stable periodic trajectories of the drilling shaft system are shown in Figs. 4–7. Figs. 4–7(a) show the comparison of the trajectories of the drilling shaft center by the reduction model and full degrees of freedom model for \( \omega = 905 \) r/min. Simultaneously, Figs. 4–7(b) are presented in contrast with the 4th order Runge–Kutta method. It is shown that the periodic response to the drilling shaft system has enough accuracy when the 12-eigenmodes model is used, and the influence of the solution method on the accuracy of results is validated. Additionally, the stable periodic of drilling shaft indicates that the drilling tool is supplied lowly by energy from whirling vibration, and the high-precision hole can be obtained in these parameters [27].

5. Numerical examples and discussion

The drilling shaft system with intermediate supports is typically a multi-part continuum with local nonlinearity. The nonlinear forces act only on the \( i \)th nodal point inside drilling hole. In order to test the method mentioned above more stably, the nonlinear dynamic responses to drilling shaft system with two intermediate supports are analyzed. Taking the drilling shaft length as \( l = 3200 \) mm, inner diameter as 19 mm, external diameter as 35 mm, drilling depth as \( l_c = 400 \) mm, the diameter of drilling hole as 28 mm, rotating speed as \( \omega = 1010 \) r/min, mass eccentricities as \( e_x = e_y = 7.2 \) \( \mu \)m (acting on all nodal point), the leading Floquet multiplier \(-1.0943\) is calculated. Based on Floquet theory [28] (a real eigenvalue on the negative real axis), the periodic trajectories of the drilling shaft is a period-doubling motion, as shown in Fig. 8. The period-doubling motion of cutting tool in drilling process implies that the work-piece surface gives a regeneration nature to the multi-lobe formation. Furthermore, because of the period-doubling motion of the drilling tool, the edge of tool repeats cutting work-piece twice, and makes the roundness error of the hole increase under the fixed direction of surface, due to the circle-land cutting more [27,29].

Taking rotating speed \( \omega \) as bifurcation parameter, the movement performance of drilling shaft system is a quasi-periodic motion, as shown in Fig. 9. Thus, the movement trajectories of the drilling shaft presented obviously the asymmetrical beat

---

**Fig. 7.** The trajectories of the drilling shaft center at #6 station for \( \omega = 905 \) r/min: (a) comparison of the trajectories of the drilling shaft center by the 12-eigenmodes model and full DOF model; (b) comparison of the trajectories of the drilling shaft center by the modified shooting method and Runge–Kutta method.

**Fig. 8.** The period-doubling trajectories of the drilling shaft center for \( \omega = 1010 \) r/min: (a) at #9 station; (b) at #7 station; (c) schematic diagram of lobing hole.
phenomenon and the sub-harmonic fractional frequency vibration for \( \omega = 1152 \text{ r/min} \). This occurs because the effect of drill wandering with the larger energy of whirling vibration reduces the movement stability of drilling shaft during the workpiece drilling, due to the fluctuation of cutting force, the disturbance of the hydrodynamic forces of cutting fluid and unbalanced forces. The quasi-periodic motion of drilling tool directly generates the instantaneous changes of cutting thickness under the hole surface, and forces the circle-land to cut more or less. So the hole profile is almost wave formation [27,30].

Taking rotating speed as \( \omega = 1531 \text{ r/min} \) and drilling depth as \( l_c = 600 \text{ mm} \), the unstable motion of the center of the drilling shaft system is shown in Fig. 10. In comparing the movement trajectories of the drilling shaft center at #9 stations with that at #7
stations, the difference of movement trajectories of the drilling shaft center is clearly visible. Considering the same rotating speed and drilling depth, only while the mass eccentricities of drilling shaft are $\varepsilon_x = \varepsilon_y = 5.1 \mu m$, the chaotic motion of the center of the drilling tool appears as shown in Fig. 11. Fig. 11(d) shows that the power spectrum of $y$ of shaft center at #9 station reveals the main characteristics of chaotic spectrum with low energy, wideband and unrepeatable. The occurrence of these phenomena above can easily lead to the unstable motion of the drilling shaft coupled bending and torsional vibration, because of the unsynchronization movement of the drilling shaft. Consequently, the drilling tool is easily subjected to be broken, worn and fractured, as well as the hole surface is scratched.

Moreover, it is seen from Figs. 8 and 9(a) that the higher rotating speed yields the bigger whirling magnitude. So the bigger roundness error will be generated. This trend is in agreement with the experimental results [29]. In addition, the regularity of the whirling motion at #9 station in Fig. 10 is better than in Fig. 11. This is because the mass eccentricity can inhibit the whirling motion of drilling shaft [10].

In the actual deep hole drilling process, the biggest concern of the machine operator is: the working state of the drill shaft is stable or unstable, and how to select the rotating speed parameters under the different drilling depths for improving the quality of hole surface. Hence, in order to guard the method mentioned above more usefully, the stable or unstable mode of the rotating state of drilling shaft are analyzed. Taking the drilling shaft length as $l=2000$ mm, inner diameter as 26 mm, external diameter as 35 mm, the diameter of drilling hole as 37 mm, mass eccentricities as $\varepsilon_x = \varepsilon_y = 13 \mu m$ acting on all nodal point and the same rotating phase angle, the stable region of the drilling tool in drilling hole process is obtained, as shown in Fig. 12. If the cutting parameters are selected in A region, the nonlinear vibration of drilling tool is the stable periodic motion. Thus, the higher-precision hole can be obtained and the working life of drilling tool is extended in this parameter area. However, if selecting the higher rotating speeds under the different drilling depth, such as in the B or C region, the motion states of drilling shaft are varied from stable periodic motion to period-doubling motion or quasi-periodic motion. Consequently, the roundness error of drilling hole becomes larger and makes the forms of hole as multi-lobe or waviness. The reason for this is the shaft whirling at higher speeds. Further, if selecting the rotational speeds in D region, the movement motion of the drilling tool is unstable motion. These can easily lead to the drilling tool broken, worn or even to the machine damaged. In the actual drilling, the parameters in D region are always not allowed to be selected. Additionally, the stable area of drilling shaft is decreasing with the drilling depths increasing. This is because the longer shaft inside hole is less rigid, and thus, a smaller disturbance from excitation forces, such as the cutting forces or the hydrodynamic forces, will easily lead to the unstable motion of drilling tool.

![Fig. 11. The chaotic trajectories of the drilling shaft center for $\omega = 1531$ r/min, $l_c = 600$ mm and $\varepsilon_x = \varepsilon_y = 5.1 \mu m$: (a) at #9 station; (b) at #7 station; (c) the time series at #9 station in the $y$ direction; (d) the power spectrum at #9 station in the $y$ direction.](image1)

![Fig. 12. The region of the rotating state of drilling tool.](image2)
6. Conclusions

A new mathematical approach is developed to calculate the nonlinear dynamic behaviors of drilling shaft with multi-span supports. The variational problem of lateral vibration of drilling shaft is deduced and the finite element model of lateral vibration of drilling shaft is established by using Timoshenko element model. The shear deformation and rotary inertia are taken into account in the model of drilling system. Mode synthesis technique with free-interface is modified to represent a reduced method of degrees-of-freedom. The practical flexible drilling system with multi-supports is used as subject to test the presented reduced method. The results show that the presented reduced method can reduce a large number of degrees-of-freedom of flexible drilling shaft system, effectively saving the computational costs under the condition of maintaining nonlinear analysis precision for the system.

The numerical schemes of this study are applied to a large-scale deep-hole drill machine with two intermediate supports. The periodic dynamic behaviors of the drilling shaft system and the stable region of rotation were detected with the present method. The results not only reveal many interesting phenomena of drilling shaft system with intermediate supports, but also show that algorithm is very stable and effective.

Acknowledgments

This work is supported by National Natural Science Foundation of China (Grant nos. 51105305 and 50705327), the Major Research Program of Shaanxi Province of China (13115 Project, Grant no. 2009DKG-25), Natural Science Foundation of Shaanxi Province of China (Grant no. 2011JQ7012) and Natural Science Foundation of Department of Education of Shaanxi Province of China (Grant nos. 2010JQ695 and 12JK680).

Appendix A

The lateral translations and rotation angles of a typical point within the element can be related to the nodal displacement vector \( \mathbf{X}^e \) which is of the form

\[
\mathbf{X}^e = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [x_1, y_1, \psi_1, \phi_1, x_2, y_2, \psi_2, \phi_2]_T
\]

By using Timoshenko shaft element model, the translational and rotational function can be written respectively as

\[
\begin{bmatrix} x(z, t) \\ y(z, t) \end{bmatrix} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & N_3 & 0 & 0 & N_4 \\ 0 & N_1 & N_2 & 0 & N_3 & N_4 & 0 & 0 \end{bmatrix} \mathbf{X}^e = [\mathbf{N}_e] \mathbf{X}^e
\]

\[
\begin{bmatrix} \psi(z, t) \\ \phi(z, t) \end{bmatrix} = \begin{bmatrix} 0 & \Gamma_1 & \Gamma_2 & 0 & \Gamma_3 & \Gamma_4 & 0 & 0 \\ \Gamma_1 & 0 & \Gamma_2 & 0 & \Gamma_3 & \Gamma_4 & 0 & 0 \end{bmatrix} \mathbf{X}^e = [\mathbf{\Gamma}_e] \mathbf{X}^e
\]

where \([\mathbf{N}_e], [\mathbf{\Gamma}_e], N_1, N_2, N_3, N_4, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4\) are written respectively as

\[
[\mathbf{N}_e] = \begin{bmatrix} N_{1x} \\ N_{2x} \end{bmatrix}, \quad [\mathbf{\Gamma}_e] = \begin{bmatrix} \Gamma_{1x} \\ \Gamma_{2x} \end{bmatrix},
\]

\[
N_1 = \frac{1}{1 + \phi} \left( 1 + \phi - \psi^2 - 2z^2 + \zeta^2 \right),
\]

\[
N_2 = \frac{1}{1 + \phi} \left( 1 + \phi - \psi^2 - 2z^2 + \zeta^2 \right),
\]

\[
N_3 = \frac{1}{1 + \phi} \left( \psi^2 + 3z^2 - 2\zeta^2 \right),
\]

\[
N_4 = \frac{1}{1 + \phi} \left( - \psi^2 - 2\phi \zeta^2 + 3z^2 \right)
\]

\[
\Gamma_1 = \frac{6}{(1 + \phi)^2} \left( \zeta^2 - \zeta^2 \right), \quad \Gamma_2 = \frac{1}{1 + \phi} \left[ 1 + \phi - (4 + \phi \zeta^2 + 3z^2) \right]
\]

\[
\Gamma_3 = \frac{6}{(1 + \phi)^2} \left( \zeta^2 - \zeta^2 \right), \quad \Gamma_4 = \frac{1}{1 + \phi} \left[ (\phi - 2\phi \zeta^2 + 3z^2) \right]
\]

where \( \phi = (12EI/kAG^2), \zeta = (z/l), E \) is the Young’s modulus, \( I \) is the moments of inertia, \( k \) is the shear coefficient, \( G \) is the shear modulus and \( A \) is the cross-sectional area of the drilling shaft.

The kinetic energy \( T^e \) of the shaft element can be given by

\[
T^e = \frac{1}{2} \dot{\mathbf{x}}^e_\mathbf{M}^e \dot{\mathbf{x}}^e + \frac{1}{2} \dot{\mathbf{X}}^e_\mathbf{M}^e \dot{\mathbf{X}}^e + \frac{1}{2} \dot{\mathbf{\omega}}^e \cdot \mathbf{S}^e \mathbf{\omega}^e
\]

where

\[
\mathbf{M}^e = \int_0^l \rho A [N_e]^T [N_e] dz, \quad \mathbf{M}^e = \int_0^l J_d [\Gamma_1]^T [\Gamma_1] dz, \quad T^e = \int_0^l \rho J_p [\Gamma_1]^T [\Gamma_1] dz
\]

\[
\rho \] is the mass density of the shaft material, \( J_d \) and \( J_p \) are the diametral and polar mass moments of inertia of shaft per unit length.

The potential energy \( V^e \) of the shaft element can be given by

\[
V^e = \frac{1}{2} \mathbf{K}^e [\mathbf{X}^e]^T \mathbf{X}^e
\]

where \( \mathbf{K}^e = [\mathbf{K}_{\text{el}}] + [\mathbf{K}_{\text{sh}}] \), \([\mathbf{K}_{\text{el}}]\) and \([\mathbf{K}_{\text{sh}}]\) are the elastic stiffness matrix and shear stiffness matrix respectively, which are of the forms

\[
[\mathbf{K}_{\text{el}}] = \int_0^l E I \left( \frac{\partial [\mathbf{\Gamma}_1]}{\partial z} \right)^T \left( \frac{\partial [\mathbf{\Gamma}_1]}{\partial z} \right) dz
\]

\[
[\mathbf{K}_{\text{sh}}] = \int_0^l \kappa AG \left[ -(\partial [N_e]_x / \partial z) - [\mathbf{\Gamma}_1] \right]^T \left[ -(\partial [N_e]_x / \partial z) - [\mathbf{\Gamma}_1] \right] dz
\]

The equations of motion for the complete drilling shaft system can be obtained by assembling the contribution of each component equation of motion, namely \( \mathbf{T} = \sum_{i=1}^{N_{\text{ele}}} \mathbf{T}_i^e \) and \( \mathbf{V} = \sum_{i=1}^{N_{\text{ele}}} \mathbf{V}_i^e \), and the final equations can be expressed in the form as

\[
\mathbf{M} \ddot{\mathbf{X}} + \mathbf{G} \mathbf{X} + \mathbf{K} \mathbf{X} = \mathbf{F}_b + \mathbf{F}_c + \mathbf{f}(\mathbf{X}, \dot{\mathbf{X}})
\]

Appendix B

The relevant calculation parameters are used:

Young’s modulus (E): 206 x 10^9 Pa
Density (\( \rho \)): 7.87 x 10^3 kg/m³
Shear modulus (G): 81 x 10^9 Pa
Dynamic viscosity of cutting fluid: 0.026 Pa s
Pressure of cutting fluid: 2 x 10^6 Pa.

References